

INVARIANT FORMS FOR CORRESPONDENCES OF CURVES

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1. Introduction

Consider the correspondences of curves defined over either an algebraically closed field or a number field. In this article, we try to classify the invariant differential forms which a correspondence admits. To state things precisely, let us recall that a correspondence in any category \mathcal{D} is a tuple $\mathbb{X} = (Y, X, \sigma_1, \sigma_2)$ where X and Y are objects in \mathcal{D} and $\sigma_1, \sigma_2 : X \rightarrow Y$ are morphisms. In our case, \mathcal{D} is the category of smooth algebraic curves defined over k , where k is either an algebraically closed curve of any characteristic or k is a number field. Let $\Omega_{K(Y)}$ denote the sheaf of rational 1-forms over the curve Y with function field $K(Y)$. $\Omega_{K(Y)}^{\otimes \nu}$ will denote its higher tensor powers of $\Omega_{K(Y)}$ for any $\nu \in \mathbb{Z}$. We would say that a form $\omega \in \Omega_{K(Y)}^{\otimes \nu}$, $\omega \neq 0$ is *invariant* of weight ν in \mathbb{X} if $\sigma_1^* \omega = \sigma_2^* \omega$. A form ω will be called *sem-invariant* if $\sigma_1^* \omega = \lambda \sigma_2^* \omega$ for some $\lambda \in k^\times$.

Now, given a correspondence \mathbb{X} , we can associate the group $\mathcal{G}_{\mathbb{X}}$ defined as

$$\mathcal{G}_{\mathbb{X}} := \{\omega \mid \omega \neq 0, \sigma_1^* \omega = \lambda \sigma_2^* \omega, \lambda \in k^\times\} / \sim$$

where the equivalence relation \sim is $\omega \sim \omega'$ if and only if $\frac{\omega}{\omega'}$ is a constant function on Y . If the degrees of the morphisms σ_1 and σ_2 are unequal, then Proposition 2.1 implies that $\mathcal{G}_{\mathbb{X}}$ is in fact free group of rank ≤ 1 . The first question that we would like to address is that, if the group $\mathcal{G}_{\mathbb{X}}$ is non-trivial, then we would try to classify the differential forms $[\omega] \in \mathcal{G}_{\mathbb{X}}$. Given a form ω on the curve Y , we define the *Conductor* of ω , denoted by $\mathcal{C}_{Y,\omega}$, to be the cardinality of the support of the divisor $\text{div } \omega$ on Y , that is $\mathcal{C}_{Y,\omega} = \#(\text{support } (\text{div } \omega))$. For example, if $Y = \mathbb{P}^1$ and $\omega = \frac{dt}{t}$ then $\text{div } \omega = -0 - \infty$ and hence $\mathcal{C}_{\mathbb{P}^1,\omega} = 2$. Our first result on general correspondences of curves \mathbb{X} is a bound on the conductor of a semi-invariant form ω in the case when the degrees of the morphisms σ_1 and σ_2 are unequal. The result says that $\mathcal{C}_{Y,\omega}$ can be uniformly bounded in terms of the genus of our projective, smooth, curves X and Y and the degrees of the morphisms σ_1 and σ_2 . It is noteworthy to mention that this bound is independent of the characteristic of the ground field k .

Theorem 1.1. *Let $\mathbb{X} = (Y, X, \sigma_1, \sigma_2)$ be a correspondence defined over an algebraically closed field k , where X and Y are projective smooth curves with*

$\deg \sigma_1 > \deg \sigma_2$, σ_1 and σ_2 are tamely ramified and let ω be a semi-invariant form. Then the conductor $\mathcal{C}_{Y,\omega}$ is bounded by a number entirely determined by the genus of X and Y and the degrees of the morphisms σ_1 and σ_2 .

$$(1.1) \quad \mathcal{C}_{Y,\omega} \leq \frac{1}{d_1 - d_2} (3(2g_X - 2) - (2d_1 + d_2)(2g_Y - 2)),$$

where $\deg \sigma_i = d_i$, for $i = 1, 2$ and g_X and g_Y denote the genus of the curves X and Y respectively.

The above theorem is only true in the case when the degree of the maps σ_1 and σ_2 are unequal. If they are equal, then the above result is false, that is, there can not be any bound on $\mathcal{C}_{Y,\omega}$. We will exhibit an example on Hecke Correspondences which shows that the conductor $\mathcal{C}_{Y,\omega}$ grows linearly with the characteristic of the ground field. Consider the Hecke correspondences [4],

$$\mathbb{X} = (X_1(1), X_1(1, l), \sigma_1, \sigma_2) \text{ over } \mathbb{Z}[1/l],$$

where $X_1(1)$ and $X_1(l)$ are modular curves of level 1 and l respectively and σ_1 and σ_2 are the degeneracy maps. Then for each prime p , one can consider the correspondence $\mathbb{X}_p = (X_1(1)_p, X_1(1, l)_p, \bar{\sigma}_{1,p}, \bar{\sigma}_{2,p})$ obtained by base changing \mathbb{X} to the algebraic closure of the residue field at the prime p , $p \nmid l$. Then for each p , there exists an invariant form ω_p such that $(\text{support } \text{div}(\omega_p)) = \text{super-singular elliptic curves [2]}$. Hence

$$\mathcal{C}_{X_1(1)_p, \omega_p} = \#\{\text{distinct roots of the Hasse polynomial}\} = \left\lfloor \frac{p}{12} \right\rfloor + 2$$

because of the irreducibility of the Hasse polynomial [6] and this shows that $\mathcal{C}_{X_1(N)_p, \omega_p}$ is unbounded. Therefore Theorem 1.1 shows us that having unequal degrees of the maps makes the conductor bounded uniformly.

We will call a non-zero form ω on Y *primitive* if its class $[\omega]$ belongs to $\mathcal{G}_{\mathbb{X}}$ and also generates $\mathcal{G}_{\mathbb{X}}$. From now on, we will consider correspondences of the projective line \mathbb{P}^1 that is we have $X = Y = \mathbb{P}^1$. We define a pair of tuples of morphisms (σ_1, σ_2) and (σ'_1, σ'_2) to be *conjugates* if there exists an automorphism φ of \mathbb{P}^1 satisfying $\sigma'_1 = \varphi \circ \sigma_1 \circ \varphi^{-1}$ and $\sigma'_2 = \varphi \circ \sigma_2 \circ \varphi^{-1}$. We define a form ω on \mathbb{P}^1 to be a *flat* form if it is either $\omega = \frac{dt}{(t-a)}$ or $\omega = \frac{(dt)^2}{(t-a)(t-b)}$ for some a and b and $a \neq b$. Note that the form $\frac{dt}{(t-a)}$ has weight 1 where as the form $\frac{(dt)^2}{(t-a)(t-b)}$ has weight 2. Let us consider a correspondence $\mathbb{X} = (\mathbb{P}^1, \mathbb{P}^1, \sigma_1, \sigma_2)$ where both σ_1 and σ_2 are tamely ramified. Our next result classifies the types of semi-invariant forms that the above correspondence \mathbb{X} can admit. However, we had to assume a technical condition of sufficient separateness between the degrees of the morphisms.

Theorem 1.2. *Let \mathbb{X} be as above and $(\sigma_1, \sigma_2) \sim (\sigma'_1, \sigma'_2)$ where σ'_1 and σ'_2 are completely ramified at a point. Also further assume that $\deg \sigma_1 \geq 14 \deg \sigma_2$. If $\mathcal{G}_{\mathbb{X}}$ is non-trivial and ω is its primitive, then ω is flat.*

Let us now consider a correspondence \mathbb{X} defined over a number field $\mathcal{F} \subset \mathbb{C}$. Let \mathfrak{p} be a place of \mathcal{F} with $k_{\mathfrak{p}}$ as its residue field and $k_{\mathfrak{p}}^a$ its algebraic closure. Hence for each place \mathfrak{p} we obtain a new correspondence $\mathbb{X}_{\mathfrak{p}} = (Y_{\mathfrak{p}}, X_{\mathfrak{p}}, \bar{\sigma}_{1,\mathfrak{p}}, \bar{\sigma}_{2,\mathfrak{p}})$ by base changing \mathbb{X} to $k_{\mathfrak{p}}^a$. Also let us again restrict ourselves to the case when $X = Y = \mathbb{P}^1$. Then by our Theorem 1.2, if for a place \mathfrak{p} , the group $\mathcal{G}_{\mathbb{X}_{\mathfrak{p}}}$ is non-trivial, then the primitive ω of $\mathcal{G}_{\mathbb{X}_{\mathfrak{p}}}$ has to be either $\omega = \frac{dt}{t-a}$, the flat form of weight 1 or $\omega = \frac{(dt)^2}{(t-a)(t-b)}$ which is the flat form of weight 2 provided that $\deg \sigma_1 \geq 14 \deg \sigma_2$. Hence the above forms play a central role as the semi-invariant forms for correspondences of \mathbb{P}^1 with unequal degrees.

Let us now look at an example. Consider $\mathbb{X} = (\mathbb{P}^1, \mathbb{P}^1, \sigma_1, \sigma_2)$ defined over \mathbb{Z} . Let $\sigma_1 = t^m$ and $\sigma_2 = t^h$ and $\omega_{\mathfrak{p}} = \frac{dt}{t}$ for all \mathfrak{p} . Then it is easy to see that for all \mathfrak{p} , $\omega_{\mathfrak{p}}$ is semi-invariant. In fact, this simply follows because $\sigma_i^* \omega_{\mathfrak{p}} = \gamma_i \omega_{\mathfrak{p}}$ for $i = 1, 2$ and some γ_i 's [1]. Now we will construct another example by “twisting” the above example. Let σ be any endomorphism of \mathbb{P}^1 . Define $\sigma_1 = \sigma \circ t^m$ and $\sigma_2 = \sigma \circ t^h$. For each \mathfrak{p} define $\omega_{\mathfrak{p}}$ as before. Then clearly $\omega_{\mathfrak{p}}$ is semi-invariant because $\sigma_1^* \omega_{\mathfrak{p}} = \sigma^*(t^m)^* \omega_{\mathfrak{p}} = \sigma^*(\gamma_1 \omega_{\mathfrak{p}}) = \gamma_1 \sigma^* \omega_{\mathfrak{p}}$ and similarly $\sigma_2^* \omega_{\mathfrak{p}} = \gamma_2 \omega_{\mathfrak{p}}$.

Our next result is the converse of the above example. It says that for a correspondence $\mathbb{X} = (\mathbb{P}^1, \mathbb{P}^1, \sigma_1, \sigma_2)$ defined over a number field \mathcal{F} and let $\mathbb{X}_{\mathfrak{p}}$ be the correspondence obtained from \mathbb{X} as described above, if the group $\mathcal{G}_{\mathfrak{p}}$ is non-trivial and is generated by primitive of weight 1 for infinitely many places \mathfrak{p} , then that determines the morphisms σ_1 and σ_2 to be the ones described in the above example.

Theorem 1.3. *Let $\mathbb{X} = (\mathbb{P}^1, \mathbb{P}^1, \sigma_1, \sigma_2)$ be defined over a number field \mathcal{F} such that $\mathcal{G}_{\mathbb{X}_{\mathfrak{p}}}$ is non-trivial and is generated by primitives of weight 1 for infinitely many places \mathfrak{p} . Then there exists an endomorphism $\sigma : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ defined over \mathcal{F} such that,*

$$(1.2) \quad (\sigma_1, \sigma_2) \sim (\lambda_1(\sigma \circ t^m), \lambda_2(\sigma \circ t^h))$$

for some integers m and h and constants λ_1 and λ_2 as in corollary 4.2.

We would now like to summarize our results and put them in context. The question that we are trying to solve was posed in [1]- classify all (σ_1, σ_2) of $\mathbb{X} = (\mathbb{P}^1, \mathbb{P}^1, \sigma_1, \sigma_2)$ defined over a number field \mathcal{F} such that $\mathcal{G}_{\mathbb{X}_{\mathfrak{p}}}$ is non-trivial for infinitely many places \mathfrak{p} . The case when σ_1 is fixed to be the identity morphism on \mathbb{P}^1 has been completely classified in [1]. Then it has been shown that the other morphism $\sigma_2 := \sigma$ can only be one of the *flat* maps- the *Multiplicative*, the *Chebyshev* and the *Latté* maps. We will briefly describe the Multiplicative and Chebyshev maps for our purpose. A map $\sigma : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is multiplicative if $\sigma(t) = t^{\pm d}$ for a positive integer d . A Chebyshev polynomial σ of degree d is the unique polynomial such that

$\sigma(t + t^{-1}) = t^d + t^{-d}$. Then for $\mathbb{X} = (\mathbb{P}^1, \mathbb{P}^1, \mathbb{1}, \sigma)$ which satisfies that $\mathcal{G}_{\mathbb{X}_{\mathfrak{p}}}$ is non trivial for infinitely many places \mathfrak{p} then

- (1) when the primitive of $\mathcal{G}_{\mathbb{X}_{\mathfrak{p}}}$ is a flat form of weight 1, then σ is conjugate to a Multiplicative function.
- (2) when the primitive of $\mathcal{G}_{\mathbb{X}_{\mathfrak{p}}}$ is a flat form of weight 2, then σ is conjugate to a Chebyshev function.

There is another possibility of the primitive other than the above two, as shown in [1], which is associated with the Lattès map. All these above three described primitives exhausts all the possibilities for the group $\mathcal{G}_{\mathbb{X}_{\mathfrak{p}}}$ to be non-trivial for infinitely many \mathfrak{p} 's where $\mathbb{X} = (\mathbb{P}^1, \mathbb{P}^1, \mathbb{1}, \sigma)$.

Now in the general case where $\mathbb{X} = (\mathbb{P}^1, \mathbb{P}^1, \sigma_1, \sigma_2)$ is defined over \mathcal{F} , if we insist that the pair of maps (σ_1, σ_2) is conjugate to the ones totally ramified at one point, say ∞ , then our Theorem 1.2 ‘morally’ says that the primitive for a non-trivial $\mathcal{G}_{\mathbb{X}_{\mathfrak{p}}}$ has to be a flat form of weight either 1 or 2. However, we do indeed assume that extra condition of $\deg \sigma_1 \geq 14 \deg \sigma_2$ to prove the result. One hopes that this condition may be done away with or else there might be interesting examples of primitives which are not flat forms in the case when $\deg \sigma_1 < 14 \deg \sigma_2$.

Our Theorem 1.3 shows that if the primitive of $\mathcal{G}_{\mathbb{X}_{\mathfrak{p}}}$ is a flat form of weight 1, then the pair of maps (σ_1, σ_2) has to come from a pair of Multiplicative functions composed with any arbitrary endomorphism σ of \mathbb{P}^1 . Hence from the above analogy listed in the case of $(\mathbb{1}, \sigma)$, we would like to conjecture that in the case when the primitive of $\mathcal{G}_{\mathbb{X}_{\mathfrak{p}}}$ is a flat form of weight 2, then our pair of morphisms (σ_1, σ_2) come from a pair of Chebyshev functions in the same manner as in the case of weight 1. This question lies as one of the motivation for future work for the author. Also the question of the third possibility of the primitive of $\mathcal{G}_{\mathbb{X}_{\mathfrak{p}}}$ remains completely open for further understanding.

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2. Proof of Theorem 1.1

Proposition 2.1. *If $\deg \sigma_1 > \deg \sigma_2$ then $\mathcal{G}_{\mathbb{X}}$ is a free group of order ≤ 1 .*

Proof of proposition 2.1. If $\mathcal{G}_{\mathbb{X}} = \{[1]\}$ then there is nothing more to prove. Hence let us assume $\mathcal{G}_{\mathbb{X}}$ is non-trivial. Suppose $[\omega], [\omega'] \in \mathcal{G}_{\mathbb{X}}$ of the same weight ν . Then $f := \frac{\omega}{\omega'} \in K(Y)$ is a rational function for Y . But since we have $\sigma_1^* \omega = \lambda \sigma_2^* \omega$ and $\sigma_1^* \omega' = \lambda' \sigma_2^* \omega'$ for some constants λ and λ' , we have

$$(2.1) \quad \sigma_1^* \left(\frac{\omega}{\omega'} \right) = \frac{\lambda}{\lambda'} \sigma_2^* \left(\frac{\omega}{\omega'} \right) \Rightarrow \sigma_1^* f = (\text{constant}) \sigma_2^* f$$

But $\deg \sigma_1^* f > \deg \sigma_2^* f$ unless f is a constant, which implies that $[\omega] = [\omega']$. This shows that for a given weight ν , there exists a unique class $[\omega] \in \mathcal{G}_{\mathbb{X}}$.

Since $\mathcal{G}_{\mathbb{X}}$ is non-trivial, there exists a form ω with the smallest positive weight μ such that $[\omega] \in \mathcal{G}_{\mathbb{X}}$. And as we have shown above, this class of weight μ is unique. Let ω' is a semi-invariant form of weight ν . then $\nu = \mu l + r$ for some integer $r < \mu$. But then $\frac{\omega'}{\omega}$ is a semi-invariant form of weight r which is a contradiction to our hypothesis for ω unless $r = 0 \Rightarrow [\omega'] = [\omega]^l$ and we are done. \square

Let $\mathbb{X} = (Y, X, \sigma_1, \sigma_2)$ be a correspondence of curves X and Y over any algebraically closed field k . Also σ_1 and σ_2 are both *tamely ramified*.

Lemma 2.2. *If ω is an invariant form of weight ν of a correspondence \mathbb{X} with $\deg \sigma_1 > \deg \sigma_2$ then,*

$$(2.2) \quad \mathcal{C}_{Y,\omega} \leq \frac{1}{\deg \sigma_1 - \deg \sigma_2} (2 \deg R_{\sigma_1} + \deg R_{\sigma_2})$$

where R_{σ_1} and R_{σ_2} are the ramification divisors of σ_1 and σ_2 respectively.

Remark 2.1. *Before proving the above lemma, we would like to remark that no assumption on the smoothness of X and Y are necessary. Neither do we need to assume any properness condition on either X and Y .*

Proof of lemma 2.2. We denote, $d_i = \deg \sigma_i$ for $i = 1, 2$. Now let us write the divisor associated to our invariant form ω as,

$$\text{div}(\omega) = \sum_{i=1}^{\mathcal{C}_{Y,\omega}} f_i y_i, f_i \in \mathbb{Z} \setminus (0), y_i \in Y$$

where $\mathcal{C}_{Y,\omega}$ is the conductor of a form ω as defined before. Then for each y_i define its pull-back via σ_1 as

$$\begin{aligned} \sigma_1^* y_i &= \sum_{j=1}^{n_i} e_{ij} \beta_{ij}, \text{ such that} \\ e_{ij} &\leq 1, \text{ when } j \leq \nu_i, \text{ for some } \nu_i \text{ and} \\ e_{ij} &\geq 2, \text{ when } j > \nu_i \end{aligned}$$

In particular, for all $j \leq \nu_i$, the points $\beta_{ij} \in X$ are unramified under the map σ_1 . Then one can write the ramification divisor for R_{σ_1} as

$$(2.3) \quad R_{\sigma_1} = \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \sum_{j=1}^{n_i} (e_{ij} - 1) \beta_{ij} + \sum_k l_k \delta_k$$

where δ_k 's are the other ramification points with ramification indices l_k in X which do not belong to the preimages of y_i 's. Hence writing out

$\text{div} (\sigma_1^* \omega) = \sigma_1^* \text{div} (\omega) + \nu R_{\sigma_1}$, we obtain,

$$\begin{aligned} \text{div} (\sigma_1^* \omega) &= \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \sum_{j=1}^{n_i} (f_i e_{ij} + \nu(e_{ij} - 1)) \beta_{ij} + \sum_k l_k \delta_k \\ &= \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \left(\sum_{j=1}^{n_i} (e_{ij}(f_i + \nu) - \nu) \right) \beta_{ij} + \sum_k l_k \delta_k \end{aligned}$$

Note that if $j \leq \nu_i$, then $\beta_{ij} \in \text{support} (\text{div} (\sigma_1^* \omega)) = \text{support} (\text{div} (\sigma_2^* \omega))$. Hence we have

$$(2.4) \quad \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \nu_i \leq |\text{support} (\text{div} (\sigma_2^* \omega))|$$

We have,

$$\begin{aligned} \mathcal{C}_{Y,\omega} d_1 &= \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \sum_{j=1}^{n_i} e_{ij} \\ &= \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \left(\sum_{j=1}^{\nu_i} 1 + \sum_{j=\nu_i+1}^{n_i} e_{ij} \right) \\ &= \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \nu_i + \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \sum_{j=\nu_i+1}^{n_i} e_{ij} \end{aligned}$$

and hence,

$$(2.5) \quad \mathcal{C}_{Y,\omega} d_1 \leq |\text{support} (\text{div} (\sigma_2^* \omega))| + \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \sum_{j=\nu_i+1}^{n_i} e_{ij}$$

From 2.3 we have,

$$\begin{aligned} \deg R_{\sigma_1} &= \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \sum_{j=\nu_i}^{n_i} (e_{ij} - 1) + \sum_k l_k \\ &\geq \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \sum_{j=\nu_i}^{n_i} e_{ij} - \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \sum_{j=\nu_i}^{n_i} 1 \\ \deg R_{\sigma_1} + \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \sum_{j=\nu_i}^{n_i} 1 &\geq \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \sum_{j=\nu_i}^{n_i} e_{ij} \end{aligned}$$

$$(2.6) \quad 2 \deg R_{\sigma_1} \geq \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \sum_{j=\nu_i}^{n_i} e_{ij}, \text{ because } \sum_{i=1}^{\mathcal{C}_{Y,\omega}} \sum_{j=\nu_i}^{n_i} 1 \leq \deg R_{\sigma_1}$$

Now $|\text{support}(\text{div}(\sigma_2^*\omega))| \leq |\text{support}(\sigma_2^*(\text{div}(\omega)))| + |\text{support}(\nu(R_{\sigma_2}))|$ which gives,

$$(2.7) \quad |\text{support}(\text{div}(\sigma_2^*\omega))| \leq \mathcal{C}_{Y,\omega}d_2 + \deg R_{\sigma_2}$$

Therefore, putting 2.5, 2.6 and 2.7 together we obtain,

$$(2.8) \quad \mathcal{C}_{Y,\omega} \leq \frac{1}{d_1 - d_2}(2\deg R_{\sigma_1} + \deg R_{\sigma_2})$$

and we are done. \square

As a corollary, we prove our first theorem,

Proof of theorem 1. Since our curves X and Y are smooth and projective with σ_1 and σ_2 tamely ramified, by Riemann-Hurwitz [7] we get

$$(2.9) \quad \deg R_{\sigma_i} = (2g_X - 2) - d_i(2g_Y - 2), \quad \forall i = 1, 2$$

Hence combining lemma 2.2 and 2.9 we obtain our result. \square

3. Proof of Theorem 1.2

We consider the restriction of our maps σ_1 , σ_2 and ω to the affine line $\mathbb{A}^1 \subset \mathbb{P}^1$ to obtain a new correspondence $\mathbb{X} = (\mathbb{A}^1, \mathbb{A}^1, \sigma_1, \sigma_2)$. We will also assume that $\deg \sigma_1 > \deg \sigma_2$.

Let $\text{div}(\omega) = \sum_{i=1}^m e_i y_i$. Then we have the following lemma,

Lemma 3.1. *If \mathbb{X} admits a semi-invariant form ω of weight ν then*

$$\sum_{i=1}^{\mathcal{C}_{\mathbb{A}^1,\omega}} e_i = -\nu$$

Proof. Since σ_i 's are endomorphisms of \mathbb{A}^1 , we note that the degree of the ramification divisor $R_{\sigma_i} = d_i - 1$ for $i = 1, 2$. Hence equating the degrees of the divisors $\text{div}(\sigma_1^*\omega)$ and $\text{div}(\sigma_2^*\omega)$ we obtain,

$$\begin{aligned} \sum_{i=1}^{\mathcal{C}_{\mathbb{A}^1,\omega}} e_i d_1 + \nu(d_1 - 1) &= \sum_{i=1}^{\mathcal{C}_{\mathbb{A}^1,\omega}} e_i d_2 + \nu(d_2 - 1), \\ \sum_{i=1}^{\mathcal{C}_{\mathbb{A}^1,\omega}} e_i d_1 + \nu d_1 - \nu &= \sum_{i=1}^{\mathcal{C}_{\mathbb{A}^1,\omega}} e_i d_2 + \nu d_2 - \nu, \\ (d_1 - d_2) \sum_{i=1}^{\mathcal{C}_{\mathbb{A}^1,\omega}} e_i &= \nu(d_2 - d_1), \\ \sum_{i=1}^{\mathcal{C}_{\mathbb{A}^1,\omega}} e_i &= -\nu \quad \square \end{aligned}$$

Lemma 3.2. *If \mathbb{X} admits a non-zero semi-invariant form ω then*

$$\mathcal{C}_{\mathbb{A}^1, \omega} \leq 2 + \frac{3(d_2 - 1)}{d_1 - d_2}$$

Proof. Since the degree of the ramification divisors R_{σ_1} and R_{σ_2} are $d_1 - 1$ and $d_2 - 1$ respectively, substituting in 2.2 we obtain our desired result. \square

Corollary 3.3. *If $d_1 \geq 4d_2$ then $\mathcal{C}_{\mathbb{A}^1, \omega} \leq 2$.*

Proof.

$$\begin{aligned} d_1 &\geq 4d_2 \\ d_1 - d_2 &\geq 3d_2 \\ \frac{1}{d_1 - d_2} &\leq \frac{1}{3d_2} \\ \frac{3(d_2 - 1)}{d_1 - d_2} &\leq \frac{3d_2 - 3}{3d_2} \\ &= 1 - \frac{1}{d_2} \\ &< 1 \\ 2 + \frac{3(d_2 - 1)}{d_1 - d_2} &< 3 \end{aligned}$$

In other words $\mathcal{C}_{\mathbb{A}^1, \omega} < 3$ but since $\mathcal{C}_{\mathbb{A}^1, \omega}$ is a natural number we conclude that $\mathcal{C}_{\mathbb{A}^1, \omega} \leq 2$. \square

Lemma 3.4. *Let \mathbb{X} be a correspondence such that $d_1 \geq 4d_2$ and admitting a semi-invariant form ω with $\mathcal{C}_{\mathbb{A}^1, \omega} = 1$, then ω is a flat form of type-1.*

Proof. Since $\mathcal{C}_{\mathbb{A}^1, \omega} = 1$ and by lemma 3.1, $\text{div}(\omega) = -\nu.y$ which is precisely a flat form of type-1. \square

Lemma 3.5. *If ω is a semi-invariant form with $\mathcal{C}_{\mathbb{A}^1, \omega} = 2$ then its weight cannot be 1.*

Proof. By lemma 3.1 we may assume that

$$(3.1) \quad \text{div } \omega = ey_1 - (e + 1)y_2, \text{ and } e \geq 1.$$

Let the pull-pack of the divisor y_2 via the two maps be,

$$(3.2) \quad \sigma_1^* y_2 = \sum_{i=1}^n e_i \beta_i \text{ and } \sigma_2^* y_2 = \sum_{j=1}^m f_j \beta'_j$$

where e_i and f_j 's are positive non-zero integers. Then one can write the ramification divisors as in (2.3) as,

$$(3.3) \quad R_{\sigma_1} = \sum_{i=1}^n (e_i - 1) \beta_i + D \text{ and } R_{\sigma_2} = \sum_{j=1}^m (f_j - 1) \beta'_j + D'$$

for some effective divisors D and D' . Hence, if we equate the poles of the divisor $\operatorname{div} \sigma_1^* \omega$ and $\operatorname{div} \sigma_2^* \omega$ we obtain,

$$\begin{aligned} \sum_{i=1}^n ((e_i - 1) - e_i(e + 1))\beta_i &= \sum_{j=1}^m ((f_j - 1) - f_j(e + 1))\beta'_j \\ \sum_{i=1}^n -(1 + e_i e)\beta_i &= \sum_{j=1}^m -(1 + f_j f)\beta'_j \end{aligned}$$

Since $-(1 + e_i e)$ and $-(1 + f_j f)$ are non-zero for any i and j implies that $n = m$, $\beta_i = \beta'_j$ and $e_i = f_j$. But then $\deg \sigma_1 = \deg \sigma_2$ which is a contradiction. \square

Theorem 3.6. *If $d_1 \geq 14d_2$ and ω is a semi-invariant form for \mathbb{X} such that $[\omega]$ is the primitive and $\mathbb{C}_{\mathbb{A}^1, \omega} = 2$, then $\operatorname{div}(\omega) = -y_1 - y_2$.*

Proof. Let $\operatorname{div}(\omega) = e.y_1 + f.y_2$ for some $e, f \in \mathbb{Z} \setminus (0)$. Then by lemma 3.1, $e + f = -\nu$ and $(e, f) = 1$ because ω is primitive. We also know that

$$\operatorname{div}(\sigma_i^* \omega) = \sigma_i^*(\operatorname{div}(\omega)) + \nu.R_{\sigma_i}, \text{ for } i = 1, 2$$

Then we have,

$$\begin{aligned} \sigma_1^*(y_1) &= \sum_{i=1}^n e_i \alpha_i \text{ and } \sigma_2^*(y_1) = \sum_{i=1}^{n'} e'_i \alpha_i \\ \sigma_1^*(y_2) &= \sum_{j=1}^m f_j \beta_j \text{ and } \sigma_2^*(y_2) = \sum_{j=1}^{m'} f'_j \beta'_j \end{aligned}$$

for some $e_i, e'_i, f_j, f'_j > 0$. We can also write the ramification divisors as-

$$\begin{aligned} R_{\sigma_1} &= \sum_{i=1}^n (e_i - 1)\alpha_i + \sum_{j=1}^m (f_j - 1)\beta_j + \sum_{k=1}^p l_k \delta_k \\ R_{\sigma_1} &= \sum_{i=1}^{n'} (e'_i - 1)\alpha'_i + \sum_{j=1}^{m'} (f'_j - 1)\beta'_j + \sum_{k=1}^p l'_k \delta'_k \end{aligned}$$

Then the divisor associated to $\sigma_1^* \omega$ is

$$\begin{aligned} \operatorname{div}(\sigma_1^* \omega) &= e \sum_{i=1}^n e_i \alpha_i + f \sum_{j=1}^m f_j \beta_j + \nu \left(\sum_{i=1}^n (e_i - 1)\alpha_i + \sum_{j=1}^m (f_j - 1)\beta_j + \sum_{k=1}^p l_k \delta_k \right) \\ &= \sum_{i=1}^n (ee_i + \nu e_i - \nu)\alpha_i + \sum_{j=1}^m (ff_j + \nu f_j - \nu)\beta_j + \sum_{k=1}^p \nu l_k \delta_k \\ &= \sum_{i=1}^n (-fe_i - \nu)\alpha_i + \sum_{j=1}^m (-ef_j - \nu)\beta_j + \sum_{k=1}^p \nu l_k \delta_k, \text{ since } e + f = -\nu \end{aligned}$$

And similarly, we have

$$\operatorname{div} (\sigma_2^* \omega) = \sum_{i=1}^{n'} (-fe'_i - \nu) \alpha'_i + \sum_{j=1}^{m'} (-ef'_j - \nu) \beta'_j + \sum_{k=1}^{p'} \nu l'_k \delta'_k$$

Claim. If any one of the factors of the form $(-fe_i - \nu)$ or $(-fe'_i - \nu)$ is 0 then $f = -1$.

Proof. If $-fe_i - \nu = 0$ for some i implies that $f = \frac{\nu}{-e_i}$. That means that f is negative because both ν and e_i are positive. By lemma 3.1 we know that $f + e = f(-e_i) = \nu$ which implies that, $e = -f(1 - e_i) \Rightarrow f \mid e$. But since ω is primitive, we have $(e, f) = 1$. Hence f can only be 1 or -1 but then f is negative and hence $f = -1$. Similar argument implies the result in the case when $-fe'_i - \nu = 0$ and this completes the proof of our claim.

Similarly we can show that,

Claim. If any one of the factors of the form $(-ef_j - \nu)$ or $(-ef'_j - \nu)$ is 0 then $e = -1$.

Now suppose ω is not a flat form. Then from the above claims, we have the following two cases to consider:

Case (1):

If none of the e and f equals -1 . Then the above claim implies that $(-fe_i - \nu), (-fe'_i - \nu) \neq 0, \forall i$ and $(-ef_j - \nu), (ef'_j - \nu) \neq 0, \forall j$ which means, all the coefficients in $\operatorname{div} (\sigma_1^* \omega)$ and $\operatorname{div} (\sigma_2^* \omega)$ are non-zero and since we have,

$$(3.4) \quad |\operatorname{support} \operatorname{div} (\sigma_1^* \omega)| = |\operatorname{support} \operatorname{div} (\sigma_2^* \omega)|$$

implies that $m + n + p = m' + n' + p'$.

Case (2): We may assume without loss of generality that $e = -1, \Rightarrow f = 1 - \nu$ and $f \neq -1$ as because then our ω is already in the required form. Since $f \neq -1$ then by the above claim, $(-fe_i - \nu) \neq 0, \forall i$ and $(-fe'_i - \nu) \neq 0, \forall i$

If Case (1): was true then from the fact,

$$\deg (\operatorname{div} (\sigma_1^* \omega)) = \deg (\operatorname{div} (\sigma_2^* \omega))$$

we obtain,

$$\begin{aligned}
\sum_{i=1}^n (-fe_i - \nu) + \sum_{j=1}^m (-ef_j - \nu) + \sum_{k=1}^p \nu l_k &= \sum_{i=1}^{n'} (-fe'_i - \nu) + \sum_{j=1}^{m'} (-ef'_j - \nu) + \sum_{k=1}^{p'} \nu l'_k \\
-f \sum_{i=1}^n e_i - \sum_{i=1}^n \nu - e \sum_{j=1}^m f_j - \sum_{j=1}^m \nu + \nu \sum_{k=1}^p l_k &= -f \sum_{i=1}^{n'} e'_i - \sum_{i=1}^{n'} \nu - e \sum_{j=1}^{m'} f'_j - \sum_{j=1}^{m'} \nu + \nu \sum_{k=1}^{p'} l'_k \\
-fd_1 - \nu n - ed_1 - \nu m + \nu \sum_{k=1}^p l_k &= -fd_2 - \nu n' - ed_2 - \nu m' + \nu \sum_{k=1}^{p'} l'_k \\
(-f - e)d_1 - \nu(n + m) + \nu \sum_{k=1}^p l_k &= (-f - e)d_2 - \nu(n' + m') + \nu \sum_{k=1}^{p'} l'_k \\
\nu d_1 - \nu(n + m) + \nu \sum_{k=1}^p l_k &= \nu d_2 - \nu(n' + m') + \nu \sum_{k=1}^{p'} l'_k \dots (*)
\end{aligned}$$

Now the right-hand side is bounded from above by

$$\begin{aligned}
\nu d_2 - \nu(n' + m') + \nu \sum_{k=1}^{p'} l'_k &\leq \nu d_2 + \nu \sum_{k=1}^{p'} l'_k \text{ (Since } \nu(n' + m') \geq 0 \text{)} \\
&\leq \nu d_2 + \nu(d_2 - 1) \text{ (Since } \sum_{k=1}^{p'} l'_k \leq d_2 - 1 \text{)} \\
&= \nu(2d_2 - 1) \dots (1)
\end{aligned}$$

Next we can find a lower bound for the left-hand side in the above inequality as follows,

$$\nu d_1 - \nu(n + m) + \nu \sum_{k=1}^p l_k \geq \nu d_1 - \nu(n + m) \text{ (Since } \nu \sum_{k=1}^p l_k \geq 0 \text{)}$$

Now

$$\begin{aligned}
n + m &\leq m + n + p \\
&= m' + n' + p' \\
&\leq d_2 + d_2 + (d_2 - 1) \text{ (Since } m' \leq d_2, n' \leq d_2, p' \leq d_2 - 1 \text{)} \\
&= 3d_2 - 1
\end{aligned}$$

And hence we get

$$\begin{aligned}
\nu d_1 - \nu(n + m) + \nu \sum_{k=1}^p l_k &\geq \nu d_1 - \nu(3d_2 - 1) \\
&= \nu(d_1 - 3d_2 + 1) \\
&\geq \nu(14d_2 - 3d_2 - 1) \text{ (Since } d_1 \geq 14d_2 \text{)} \\
&= \nu(11d_2 - 1) \dots (2)
\end{aligned}$$

Therefore combining (1) and (2) we get

$$\nu d_1 - \nu(n+m) + \nu \sum_{k=1}^p l_k \geq \nu(11d_2 - 1) \not\geq \nu(2d_2 - 1) \geq \nu d_2 - \nu(n' + m') + \nu \sum_{k=1}^{p'} l'_k$$

which contradicts (*) and hence removes the possibility of Case (1) for ω .

Now we proceed to show that Case (2) is also not possible for ω . In this situation as discussed above, we have $e = -1$ and $f = 1 - \nu \neq -1$.

For any divisor $H = \sum a_i P_i$, we define $|H| = \sum |a_i|$. Then consider the divisor

$$D = \sum_{i=1}^n (-f e'_i - \nu) \alpha_i + \sum_{k=1}^{p'} \nu l'_k \delta_k$$

Then we have, $\deg D \leq |D| \leq |\operatorname{div} \sigma_1^* \omega| = |\operatorname{div} \sigma_2^* \omega|$.

$$\begin{aligned} \deg D &\leq \sum_{i=1}^{n'} |(-f e'_i - \nu)| + \sum_{j=1}^{m'} |(-e f'_j - \nu)| + \sum_{k=1}^{p'} |\nu l'_k| \\ &\leq \sum_{i=1}^{n'} (|-f e'_i| + |\nu|) + \sum_{j=1}^{m'} (|-e f'_j| + |\nu|) + \sum_{k=1}^{p'} \nu l'_k \quad (\text{Since } l'_k \geq 0, \forall k) \end{aligned}$$

Since $f < 0$ we get that $-f e'_i > 0$, $\forall i$ and hence $|-f e'_i| = -f e'_i$. and also by the same reason we obtain $|-e f'_j| = -e f'_j$, $\forall j$. And $\nu > 0$, our above inequality becomes:

$$\begin{aligned} \deg D &\leq \sum_{i=1}^{n'} (-f e'_i + \nu) + \sum_{j=1}^{m'} (-e f'_j + \nu) + \sum_{k=1}^{p'} \nu l'_k \quad (\text{Since } l'_k \geq 0, \forall k) \\ &= -f \sum_{i=1}^{n'} e'_i + \sum_{i=1}^{n'} \nu - e \sum_{j=1}^{m'} f'_j + \sum_{j=1}^{m'} \nu + \nu \sum_{k=1}^{p'} l'_k \\ &= -f d_2 + \nu(n' + m') - e d_2 + \nu \sum_{k=1}^{p'} l'_k \\ &= (-e - f) d_2 + \nu(n' + m') + \nu \sum_{k=1}^{p'} l'_k \\ &\leq \nu d_2 + 2d_2 \nu + \nu(d_2 - 1) \quad (\text{Since } n', m' \leq d_2 \text{ and } \sum_{k=1}^{p'} l'_k \leq d_2 - 1) \\ &= \nu(d_2 + 2d_2 + d_2 - 1) \\ &= \nu(4d_2 - 1) \quad (3) \end{aligned}$$

Also, considering the degree of the divisor D we obtain,

$$\begin{aligned}
\deg D &= \sum_{i=1}^n (-fe_i - \nu) + \sum_{k=1}^p \nu l_k \\
&= -f \sum_{i=1}^n e_i - \sum_{i=1}^n \nu + \nu \sum_{k=1}^p l_k \\
&\geq -fd_1 - \nu n \quad (\text{Since } \sum_{k=1}^p l_k \geq 0) \\
&= (\nu - 1)d_1 - \nu n
\end{aligned}$$

Here we note that $n \leq n' + m' + p'$ since none of the $(-fe_i - \nu)$'s are zero and the right hand side of the inequality is the size of the support of the divisor $\text{div}(\sigma_2^* \omega)$ and we know $n', m' \leq d_2$ and $p' \leq d_2 - 1$. Hence we get that $n \leq 3d_2 - 1$. Substituting this in the above inequality we obtain,

$$\begin{aligned}
\deg D &\geq (\nu - 1)d_1 - \nu(3d_2 - 1) \\
&= \nu \left(\left(1 - \frac{1}{\nu}\right) d_1 - 3d_2 + 1 \right)
\end{aligned}$$

By lemma 3.5, $\nu \geq 2 \Rightarrow \frac{1}{\nu} \leq \frac{1}{2} \Rightarrow -\frac{1}{\nu} \geq -\frac{1}{2} \Rightarrow 1 - \frac{1}{\nu} \geq 1 - \frac{1}{2} = \frac{1}{2}$. Hence we have

$$\deg D \geq \nu \left(\frac{1}{2} d_1 - 3d_2 + 1 \right)$$

But by our hypothesis, $d_1 \geq 14d_2$ and we get,

$$\deg D \geq \nu \left(\frac{1}{2} \cdot 14d_2 - 3d_2 + 1 \right) = \nu(4d_2 + 1) \quad (4)$$

We combine (3) and (4) to obtain

$$\deg D \geq \nu(4d_2 + 1) \not\geq \nu(4d_2 - 1) \geq \deg D$$

and here lies the contradiction for Case (2). \square

Proof of theorem 1.2. By corollary 3.3 we need to check for the two cases when $\mathcal{C}_{\mathbb{A}^1, \omega}$ is either 1 or 2. Lemma 3.4 shows that when $\mathcal{C}_{\mathbb{A}^1, \omega} = 1$ then $\omega = \frac{1}{t-y} dt$, which implies that weight of ω is 1 and $\text{div } \omega = -y - \infty$.

And when $\mathcal{C}_{\mathbb{A}^1, \omega} = 2$, theorem 3.6 shows that $\omega = \frac{1}{(t-y_1)(t-y_2)} (dt)^2$, which implies that the weight of ω is 2 and $\text{div } \omega = -y_1 - y_2$. And this ends the proof. \square

4. Proof of theorem 1.3

Call $\tilde{\omega} = \sigma_1^* \omega = \sigma_2^* \omega$. Let $x \in X$ then by [1],

$$(4.1) \quad \text{ord}_x(\tilde{\omega}) + \nu = e_{\sigma_i}(x)(\text{ord}_{\sigma_i(x)} \omega + \nu) \text{ for } i = 1, 2$$

If ω is a flat form of weight 1, it is easy to see that there exists an automorphism of \mathbb{P}^1 fixing ∞ such that $\sigma'_1 \sim \sigma_1$ and $\sigma'_2 \sim \sigma_2$ and $\frac{dt}{t}$ is a semi-invariant form for (σ'_1, σ'_2) . Hence it is sufficient to assume that the semi-invariant form ω is of the form $\frac{dt}{t}$.

Proposition 4.1. *When $\nu = 1$, then $\text{support } \sigma_1^*(\text{div } \omega) = \text{support } \sigma_2^*(\text{div } \omega)$.*

Proof. Let ω be the primitive semi-invariant form associated to $\mathcal{C}_{\mathbb{A}^1, \omega} = 1$. Then $\text{div } (\omega) = -0 - \infty \in \text{div } (\mathbb{P}^1)$. Then,

$$\begin{aligned} y \in \text{support } \sigma_1^*(\text{div } \omega) &\iff \sigma_1(y) \in \text{support } \text{div } \omega \\ &\Rightarrow \text{ord}_{\sigma_1(y)} \omega = -1 \\ \Rightarrow e_{\sigma_2(y)}(\text{ord}_{\sigma_2(y)} \omega + 1) &= 0, \text{ because of 4.1} \\ &\Rightarrow \text{ord}_{\sigma_2(y)} \omega = -1 \text{ since } e_{\sigma_2}(y) \geq 1 \end{aligned}$$

which means that $y \in \text{support } \sigma_2^*(\text{div } \omega) \Rightarrow \text{support } \sigma_1^*(\text{div } \omega) \subset \text{support } \sigma_2^*(\text{div } \omega)$ and similarly we can show for the other direction and that concludes our proof. \square

Let $S = \{\alpha_1, \dots, \alpha_n\} = \text{support } \sigma_i^*\{0\}$ for $i = 1, 2$.

Corollary 4.2. *We have the following expressions for our σ_1 and σ_2 ,*

$$(4.2) \quad \sigma_1(t) = \lambda_1 \prod_{i=1}^n (t - \alpha_i)^{e_i} \text{ and } \sigma_2(t) = \lambda_2 \prod_{i=1}^n (t - \alpha_i)^{f_i}$$

for some tuple of positive non-zero integers (e_1, \dots, e_n) and (f_1, \dots, f_n) and λ_1 and λ_2 are constants in the ground ring.

Proof. This follows from proposition 4.1. Since both σ_1 and σ_2 are totally ramified at ∞ , all the elements of S are all the roots of the polynomials describing the maps σ_1 and σ_2 respectively. Hence we obtain our required factorisation of both σ_1 and σ_2 and we are done. \square

Let k be a field of characteristic p . Then there exists a group homomorphism $\theta_p : \mathbb{Z}_{(p)}^\times \rightarrow k^\times$ defined by $\theta_p\left(\frac{m}{n}\right) = \frac{\bar{m}}{\bar{n}}$ where \bar{m} denotes the reduction of m modulo p .

Lemma 4.3. $\ker \theta_p = \left\{ \frac{m}{n} \mid m \equiv n \pmod{p} \right\}$

Proof. $\frac{m}{n} \in \ker \theta_p$ iff $\frac{\bar{m}}{\bar{n}} = 1$ iff $\bar{m} \equiv \bar{n}$ and we are done. \square

Let \mathbb{X} now be a correspondence defined over a number field \mathcal{F} . Let p be the characteristic of the residue field at \mathfrak{p} .

Lemma 4.4. *If $(2\deg \sigma_1)(\deg \sigma_2) < p$ then there exists a function σ defined over \mathcal{F} such that,*

$$\sigma_1(t) = \lambda_1(\sigma(t))^m \text{ and } \sigma_2(t) = \lambda_2(\sigma(t))^h$$

for some non-negative integers m and h and λ_1 and λ_2 as in corollary 4.2. In other words, both σ_1 and σ_2 are a composition of σ and a multiplicative function $\lambda_1 t^m$ or $\lambda_2 t^h$ respectively.

Proof.

$$\begin{aligned}\sigma_1^*(\omega) &= l\sigma_2^*(\omega), \text{ for some } l \\ \frac{\sigma_1'}{\sigma_1} &= l \frac{\sigma_2'}{\sigma_2} \\ \sum_{i=1}^n \frac{e_i}{t - \alpha_i} &= l \sum_{i=1}^n \frac{f_i}{t - \alpha_i} \text{ by corollary 4.2}\end{aligned}$$

Note that $(e_i, p) = (f_i, p) = 1$ for all i because $e_i < d_1 < p$ and $f_i < d_2 < p$. Hence the above equality is possible only if $\theta_p(\frac{e_i}{f_i}) = \theta_p(\frac{e_j}{f_j}) \equiv l$ for all i and j , i.e. $\frac{e_i f_j}{f_i e_j} \in \ker \theta_p$. By lemma 4.3 we have $e_i f_j - f_j e_i = pg$ for some integer g . We claim that $g = 0$. If not then $|e_i f_j - f_j e_i| = |p||g| > |p|$. On the other hand, $|e_i f_j - f_j e_i| \leq |e_i f_j| + |e_j f_i| \leq 2d_1 d_2$ but $2d_1 d_2 < p$ which is a contradiction and proves our claim. In other words we have $\frac{e_i}{f_i} = \frac{e_j}{f_j} = \frac{m}{h}$ for all i and j and for some integers m and h with $(m, h) = 1$. Hence we have $e_i h = f_i m$ for all i . Take $g_i = e_i/m = f_i/h$ and set $\sigma(t) = \prod_{i=1}^n (t - \alpha_i)^{g_i}$. Then one checks easily that $\sigma_1(t) = \lambda_1(\sigma(t))^m$ and $\sigma_2(t) = \lambda_2(\sigma(t))^h$ and we are done. \square

Proof of theorem 1.3. Since \mathbb{X} over the number field \mathcal{F} admits the flat form for infinitely many places, chose a \mathfrak{p} such that its residue field has characteristic $p > 2\deg \sigma_1 \deg \sigma_2$. Then the e_i 's and $f_i \in \mathbb{Z}_{(p)}^\times$ and the theorem follows. \square

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